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Completeness of the Alexandrov Topology for Space-Time

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We show that the Alexandrov topology for a subset X of a space-time with nondegenerate space-time metric is complete iff X is strongly causal. Therefore, the property of the Alexandrov topology being complete and the property of being Hausdorff coincide. There is thus no physically measurable distinction between the Hausdorff nature and the completeness of the Alexandrov topology for space-time.

1. INTRODUCTION

This note proves two points of interest concerning the Alexandrov topology for a space-time (V, g): (i) the Alexandrov topology for a subset $X \subset V$ on which the space-time metric g is nondegenerate at each point is T_1 completely regular iff X is strongly causal. Hence the strongly causal subset of a space-time is a pseudometric subspace; in fact, the Alexandrov topology is metric iff X is strongly causal; (ii) the Alexandrov topology for a subset $X \subset V$ (same conditions on X) is complete iff X is strongly causal. Thus on the strongly causal subset X, we have the equivalences

$$\begin{aligned} X \text{ is strongly causal} &\Leftrightarrow T_A(X) \text{ is Hausdorff} \\ &\Leftrightarrow T_A(X) \text{ is } T_1 \text{ completely regular} \\ &\Leftrightarrow T_A(X) = T_{\text{man}}(X) \Leftrightarrow T_A(X) \text{ is complete} \\ &\Leftrightarrow T_A(X) \text{ is metric} \end{aligned}$$

where $T_A(X)$ and $T_{\text{man}}(X)$ are the Alexandrov topology and the manifold topology, respectively, for X.

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The significance of these results are threefold. First, it has long been known (Kronheimer and Penrose, 1967; Penrose, 1972) that the causal properties of a space-time are global in nature, and that the manifold topology of the strongly causal region can be determined by observation of causal relationships since it coincides with the Alexandrov topology on such regions. It follows that any property of the manifold topology, such as convergence of filters or sequences, has an analogous property written in terms of causal relations on the strongly causal region. The present work is a translation of the T_1 complete regularity and the completeness of the Alexandrov topology into statements of causal properties.

Second, the present work shows that there is no physically measurable difference between the property of the Alexandrov topology of being Hausdorff and that of being complete. The manifold topology naturally inherits this property on the strongly causal region. This result holds significance for a theory of topology change in space-time, for it is necessary to know when two or more global properties coincide, hence are equivalent in the context of a theory of measurement.

Third, since strong causality is equivalent to completeness of the Alexandrov topology, it follows that incompleteness of the Alexandrov topology is not an adequate criterion of space-time singularity except in the crude sense of indicating a violation of strong causality. We show that incompleteness of the manifold topology is a similarly inadequate criterion of singularity.

2. PROPERTIES OF THE ALEXANDROV TOPOLOGY

We will define a space-time to be a pair (V, g), where V is a set equipped with the manifold topology $T_{\text{man}}(V)$ which is assumed to be Hausdorff, connected, oriented, and pseudo-Riemannian. The quantity g is a space-time metric, that is, a pseudo-Riemannian, locally Lorentzian solution of Einstein's equations that is assumed to be nondegenerate at each point of V, hence $T_{\text{man}}(V)$ is paracompact since it is Hausdorff (Geroch, 1968). In the following, Y will denote an arbitrary topological space, and the term "metric" will denote a topological metric and not a space-time metric.

We say that a space (Y, T), where $T(Y)$ is a topology for Y, is uniformizable if a separating uniformity exists that is compatible with the topology T. It is well known that a space is uniformizable iff it is T_1 completely regular (Dugundji, 1966). Note that we require the uniformity to be separating (which is not part of the general definition) to ensure the uniformity-compatible topology is Hausdorff. Finally, a metric topology is said to be complete if a complete metric d exists that induces the topology, i.e., a metric d that induces the topology and for which every d-Cauchy sequence converges.

A space-time (V, g) is said to be strongly causal at a point $p \in V$ if each neighborhood of p, $U(p)$, contains a neighborhood $W(p)$ such that no nonspacelike curve (in the pseudo-Riemannian geometry determined by g) originating at p intersects $W(p)$ more than once. Let x be a point in a space-time (V, g). Then the chronological future of x in V, denoted by $I^+(x)$, is the set of points that can be reached in V from x by future-directed timelike curves. The chronological past of x, $I^-(x)$, is defined similarly using past-directed timelike curves. If for each pair x, $Y \in V$, we form the intersection $U_{xy} = I^+(x) \cap I^-(y)$, then the family $\mathfrak{U} = \{U_{xy} | x, y \in V\}$ is a basis for the Alexandrov topology for V (Kronheimer and Penrose, 1967). We will make extensive use of the following equivalence relation for a subset $X \subset V$:

$$X \text{ is strongly causal} \Leftrightarrow T_A(X) \text{ is Hausdorff}$$
$$\Leftrightarrow T_A(X) = T_{\text{man}}(X)$$

where $T_A(X)$ is the Alexandrov topology for X induced by the Alexandrov topology for V, and $T_{\text{man}}(X)$ is the manifold topology for X induced by the manifold topology for V.

The conditions under which the Alexandrov topology is T_1 completely regular, hence uniformizable, are easily established. Using the symbols $T_A(X)$ and $T_{\text{man}}(X)$ as defined above, the basic result is the following:

Theorem 1. Let (V, g) be a space-time. Then for any subset $X \subset V$, $T_A(X)$ is T_1 completely regular iff X is strongly causal.

Proof. Let SC denote "X is strongly causal," and T_1CR denote "$T_A(X)$ is T_1 completely regular." (i) SC $\Rightarrow T_1$CR: SC $\Leftrightarrow T_A(X) = T_{\text{man}}(X)$. $T_{\text{man}}(X)$ is Hausdorff and paracompact, hence T_1 normal, hence T_1 completely regular. (ii) T_1CR $\Rightarrow$ SC: Each T_1 completely regular topology is Hausdorff $\Rightarrow T_A(X)$ is Hausdorff $\Leftrightarrow$ SC. ■

The result for the completeness of the Alexandrov topology is as follows:

Theorem 2. Let (V, g) be a space-time. Then for any subset $X \subset V$, $T_A(X)$ is complete iff X is strongly causal.

Proof. Let T_A denote "$T_A(X)$ is complete," and SC denote "X is strongly causal." (i) SC $\Rightarrow T_A$: SC $\Leftrightarrow T_1$CR (Theorem 1) $\Leftrightarrow T_A(X)$ is uniformizable. SC $\Leftrightarrow T_A(X) = T_{\text{man}}(X) \Rightarrow T_A(X)$ is paracompact. Each paracompact uniformizable topology is complete (Page, 1978)$\Rightarrow T_A(X)$ is complete. (ii) $T_A \Rightarrow$SC: If $T_A(X)$ is complete, then by definition $T_A(X)$ is metric, hence Hausdorff, hence SC. ■

Corollary. $T_A(X)$ complete $\Rightarrow T_{\text{man}}(X)$ complete.

Proof. $T_A(X)$ complete $\Leftrightarrow X$ is strongly causal $\Leftrightarrow T_A(X) = T_{\text{man}}(X)$. ■

We note that the converse of this corollary is false. The static Einstein universe (Ryan and Shepley, 1975) and Misner's example of a pseudo-Riemannian metric on the 2-Torus (Misner, 1963) are not strongly causal since the manifold topology is compact, hence the Alexandrov topology is not complete. But the manifold topology is complete in both cases since they are compact Riemannian manifolds.

3. DISCUSSION

One might expect that a space-time with an incomplete Alexandrov topology or incomplete manifold topology would be singular in the sense that any metric d chosen to induce the topology would possess nonconvergent d-Cauchy sequences, hence the space-time event set might be "missing points."

Theorem 2 shows this apparently reasonable conjecture is false for the Alexandrov topology since completeness of the Alexandrov topology is equivalent to strong causality, hence is not in any normal sense a criterion of singularity. Conversely, completeness of the Alexandrov topology is not a criterion of nonsingularity, as there exist many space-times that are conventionally regarded as singular which have complete Alexandrov topologies in the nondegenerate region. Two examples are the Schwarzchild space and the Friedmann–Robertson–Walker cosmologies, which are well known to be strongly causal, hence have complete Alexandrov topologies. The incompleteness of the manifold is similarly deficient as a criterion of singularity. Taking the contrapositive of the Corollary and using Theorem 2 gives

$$T_{\text{man}}(X) \text{ incomplete} \Rightarrow X \text{ not strongly causal}$$

Hence each space-time with incomplete manifold topology violates strong causality.

We note that only the Hausdorff nature of a complete topology was used in the proof of Theorem 2. We specifically neglected the convergence properties of a complete topology. If we select a complete metric space uniformly isomorphic to the strongly causal subset, then the convergence of specific topological structures in the complete metric topology carries over exactly into the Alexandrov topology. In this manner, statements about the convergence of a topological structure in the complete metric space could be translated into equivalent statements about the strongly causal subset. This could yield new properties of the strongly causal region of a space-time.

Theorem 2 has inplications for a theory of topology change in space-time. Geroch (1967) showed that in a compact Lorentzian 4-manifold V whose boundary is the disjoint union of two compact spacelike 3-manifolds S, S', which permits a continuous choice of the forward light cone at each point, and which contains no closed, timelike curves, then S and S' are diffeomorphic. Hence, topology change in V implies either that a continuous choice of the forward light cone cannot be made (e.g., topology change also reverses the direction of time) and/or V contains closed timelike curves. Applying Theorem 2 to the latter possibility shows $T_A(V)$ is incomplete. One might then expect that an additional topology change, namely, the Cantor completion of $T_A(V)$, might resolve the dilemma by producing a space-time V' with complete Alexandrov topology, hence V' is strongly causal, hence contains no closed timelike curves. This would be a contradiction of Geroch's result.

This apparent contradiction can be resolved by noting that $T_A(V)$ is metric iff $T_A(V)$ is complete [viz., $T_A(V)$ is metric by definition if it is complete; conversely, if $T_A(V)$ is metric, it is Hausdorff, hence strongly causal, hence complete]. It is therefore meaningless to speak of the Cantor completion of the Alexandrov topology, and we cannot manufacture strongly causal space-times by the Cantor completion of the Alexandrov topology of a strong causality-violating space-time.

There are two other questions of interest related to the present work. First, it would be desirable to determine the homeomorphy classes of space-time metrics in the Alexandrov topology. This point also has relevance for a theory of topology change in space-time for it is necessary to have a systematic method to determine when two space-time metrics result in homeomorphic Alexandrov topologies.

Finally, if the equivalent of strong causality is completeness of the Alexandrov topology, what is the topological equivalent of stable causality? Since stable causality is a more reasonable causality criterion for quantum space-times than is strong causality (Hawking and Ellis, 1973), it would be interesting and useful to know the equivalent topological concept.

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